Gosford High School

2022 **TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Name and Student Number on the Question Writing Booklet provided

Total Marks: 70

Section I – 10 marks (pages 2 – 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7 – 11)

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks

Attempt Questions 1 – 10.

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10.

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1 What is the derivative of $\tan^{-1}(x^4)$ with respect to x?

A. $\dfrac{1}{1+x^8}$

B. $\dfrac{4x^3}{1+x^8}$

C. $\dfrac{4x^3}{1+x^4}$

D. $\dfrac{x^4}{1+x^8}$

2 Given $\underset{\sim}{a} = 5\underset{\sim}{i} + 5\underset{\sim}{j}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} - 2\underset{\sim}{j}$, which of the following represents $proj_{b}a$?

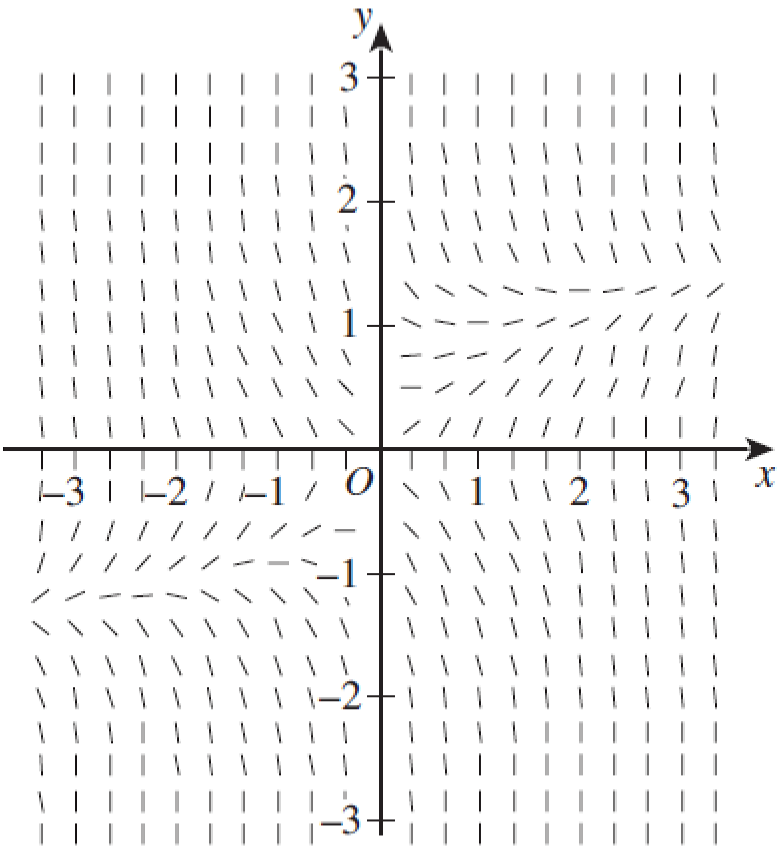
A. $40\underset{\sim}{i} - 20\underset{\sim}{j}$

B. $20\underset{\sim}{i} - 10\underset{\sim}{j}$

C. $2\underset{\sim}{i} - \underset{\sim}{j}$

D. $\dfrac{4}{5}\underset{\sim}{i} - \dfrac{2}{5}\underset{\sim}{j}$

3 The slope field for a first order differential equation is shown.



Which of the following best represents the differential equation shown in the slope field?

A. $\dfrac{dy}{dx} = \dfrac{x}{y} - y^2$

B. $\dfrac{dy}{dx} = \dfrac{x}{y} + y^2$

C. $\dfrac{dy}{dx} = -\dfrac{x}{y} - y^2$

D. $\dfrac{dy}{dx} = -\dfrac{x}{y} + y^2$

4 Which of the following integrals is obtained when the substitution $u = (\ln x)^2$ is applied to $\int_e^{e^2} \frac{(\ln x)^3}{x} dx$?

A. $\frac{1}{2}\int_1^4 u\, du$

B. $2\int_1^4 u\, du$

C. $2\int_1^4 u^{\frac{3}{2}}\, du$

D. $\int_1^4 u^6\, du$

5 Which integral finds the volume of the solid formed when $y = \sin x$ is rotated around the x-axis between $x = 0$ and $x = \frac{\pi}{3}$?

A. $\frac{\pi}{2}\int_0^{\frac{\pi}{3}} \cos 2x - 1\, dx$

B. $\frac{\pi}{2}\int_0^{\frac{\pi}{3}} 1 - \cos 2x\, dx$

C. $\frac{\pi}{2}\int_0^{\frac{\pi}{3}} 1 + \cos 2x\, dx$

D. $\pi\int_0^{\frac{\pi}{3}} \cos 2x - 1\, dx$

6 A Year 12 Biology class consists of 10 girls and 15 boys.

In how many ways can a group of three boys and two girls be chosen from this class for a group to work on an investigation project?

A. 250

B. 900

C. 12 600

D. 20 475

7 What is the domain and range of the function $y = 6\cos^{-1}(3x)$?

A. Domain $[0, 6\pi]$; Range $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

B. Domain $\left[-\frac{1}{3}, \frac{1}{3}\right]$; Range $[0, 3\pi]$.

C. Domain $\left[-\frac{1}{3}, \frac{1}{3}\right]$; Range $[0, 6\pi]$.

D. Domain $[0, 3\pi]$; Range $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

8 A curve is described by the parametric equations below.

$x = \frac{t}{2}$ $\quad y = 3t^2$

What is the cartesian equation of the curve?

A. $x = 12y^2$

B. $y = \frac{3x^2}{4}$

C. $y = \frac{2}{x}$

D. $y = 12x^2$

9 Given that α and β are both acute angles, evaluate $\sin(\alpha+\beta)$ if $\sin\alpha = \frac{8}{17}$ and $\sin\beta = \frac{4}{5}$

A. $\frac{108}{85}$

B. $\frac{84}{85}$

C. $\frac{36}{85}$

D. $\frac{28}{85}$

10 The expression $2\cos x - 3\sin x$ is written in the form $R\cos\left(x+\theta\right)$, where $R > 0$ and $0 \le \theta \le \frac{\pi}{2}$. What is the value of $\tan\theta$?

A. $-\frac{3}{2}$

B. $-\frac{2}{3}$

C. $\frac{2}{3}$

D. $\frac{3}{2}$

Section II

60 marks
Attempt Questions 11 – 14.
Allow about 1 hour and 45 minutes for this section.

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (**14 marks**) Use the Question 11 writing booklet.

(a) Show that $y = Ae^{3x} + Be^{-2x}$ satisfies the differential equation **2**

$y'' - y' - 6y = 0$ for any real values of A and B.

(b) When the polynomial $P(x)$ is divided by $x^2 - 1$ the remainder is $3x - 1$. **2**

What is the remainder when $P(x)$ is divided by $x - 1$?

(c) Differentiate $\log_e\left(\cos^{-1} x\right)$ **2**

(d) Using the substitution $u^2 = x + 1$, where $u > 0$, find $\int_0^3 \frac{x+2}{\sqrt{x+1}}\,dx$ **3**

(e) Solve the inequality $\frac{2x}{x-1} \geq 1$ **3**

(f) Find the equation of the curve $f(x)$ that passes through the point $\left(0\,,\ -\frac{\pi}{2}\right)$ and has **2**

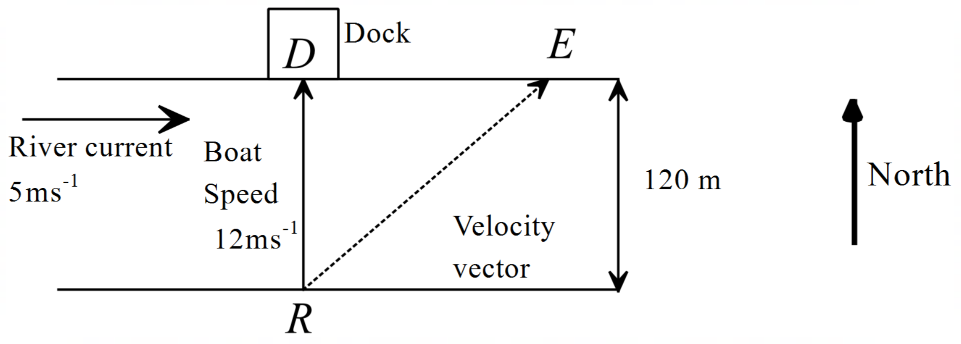
$$f\,'(x) = -\frac{3}{\sqrt{16 - 9x^2}}\,.$$

End of Question 11

Question 12 (**16 marks**) Use the Question 12 writing booklet.

(a) Rylie has a boat which moves at a top speed of 12 ms^{-1} in still water.

From point R, he wants to go due north to point D on the opposite side of the river, as shown in the diagram below.



Today the current in the river is flowing at 5 ms^{-1}. From R, he steers the boat due north toward D at top speed. Due to the current he drifts down the river and arrives at point E.

(i) Taking R as the origin, write down Rylie's velocity vector, $\underset{\sim}{v}$, in the form $\underset{\sim}{v} = x\underset{\sim}{i} + y\underset{\sim}{j}$ and find the magnitude of this vector. **2**

(ii) What is the bearing, to the nearest degree, of Rylie's velocity vector and how far does he travel from R to E? **2**

(iii) On what bearing should Rylie have pointed the boat, so that he arrived at the dock, D, with the boat travelling at a speed of 12 ms^{-1}? **2**

(b) Find $\int \sin^2 x \cos^2 x \, dx$ **2**

(c) Consider the graph of $f(x) = 2x\sin^{-1} x$, where $-1 \le x \le 1$. **2**

(i) Show that $f(x)$ is an even function. **1**

(ii) Hence, sketch the graph of $f(x)$, showing all important features including the intercept(s) and endpoint(s). **2**

(d) Prove by mathematical induction, that for all integers $n \ge 1$, **3**

$$1 + 2\times 2^{-1} + 3\times 2^{-2} + 4\times 2^{-3} + \ldots + n\times 2^{-(n-1)} = \frac{2^{n+1} - n - 2}{2^{n-1}}$$

End of Question 12

Question 13 (14 marks) Use the Question 13 writing booklet.

(a) (i) Show that $\cos(A-B)=\cos A\cos B(1+\tan A\tan B)$ **1**

(ii) Suppose that $0<B<\dfrac{\pi}{2}$ and $B<A<\pi$. **1**

Deduce that if $\tan A\tan B=-1$, then $A-B=\dfrac{\pi}{2}.$

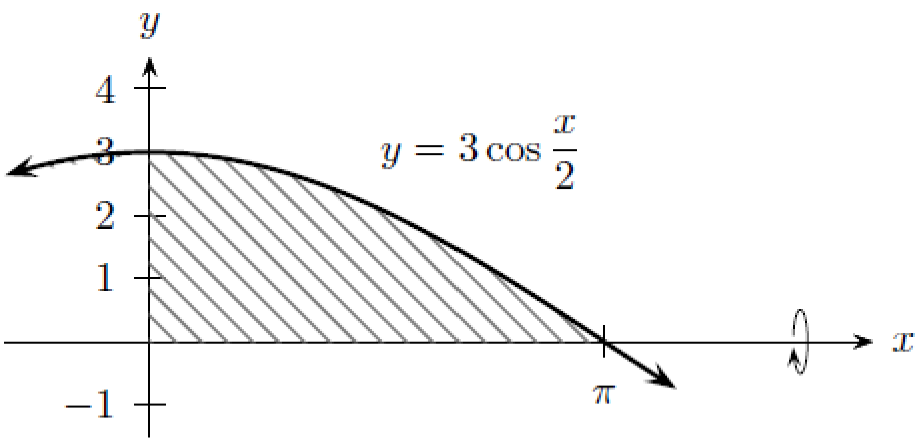
(b) Solve $\sqrt{3}\sin\theta+\cos\theta-\sqrt{3}=0$ for $\theta\in[0,2\pi]$ **3**

(c) A projectile P is fired from level ground at $30\,\text{m/s}$, at an angle of 60°. By considering the equations $\ddot{x}=0$ and $\ddot{y}=-g$, find the position vector after t seconds. Assume that $g=10\,\text{m/s}^2$. **3**

(d) Find the solution for the following differential equation: **3**

$$\frac{dy}{dx}=\sqrt{1-y^2}\text{, if } y\left(\frac{5\pi}{6}\right)=-\frac{1}{2}$$

(e) The region bounded by the graph $y=3\cos\left(\dfrac{x}{2}\right)$ and the x-axis between $x=0$ and $x=\pi$ is rotated about the x-axis to form a solid. **3**



Find the exact volume of the solid.

End of Question 13

Question 14 (**16 marks**) Use the Question 14 writing booklet.

(a) Megan has taken a frozen quiche from the freezer with an initial temperature $T = 0^{\circ}$ C and placed it in an oven set to 200° C. After ten minutes she checks the quiche and now it has a temperature of $T = 20^{\circ}$ C. The rate that the temperature of the quiche increases is proportional to the difference in temperature between the oven and the quiche.

(i) Write a differential equation to model this scenario, where T is the temperature of the quiche and t is the time in minutes, and use it to show that $T = 200 + Ae^{-kt}$. **2**

(ii) Find the value of k in exact form. **2**

(iii) The quiche is ready when the internal temperature reaches 60° C. How much longer after Megan checked the quiche will it be ready? Answer to the nearest minute. **2**

(b) It is given that $P(x) = (x-a)^3 + (x-b)^2$. The remainder when $P(x)$ is divided by $(x-b)$ is -8.

(i) Show that when $P(x)$ is divided by $(x-a)$, the remainder is 4. **2**

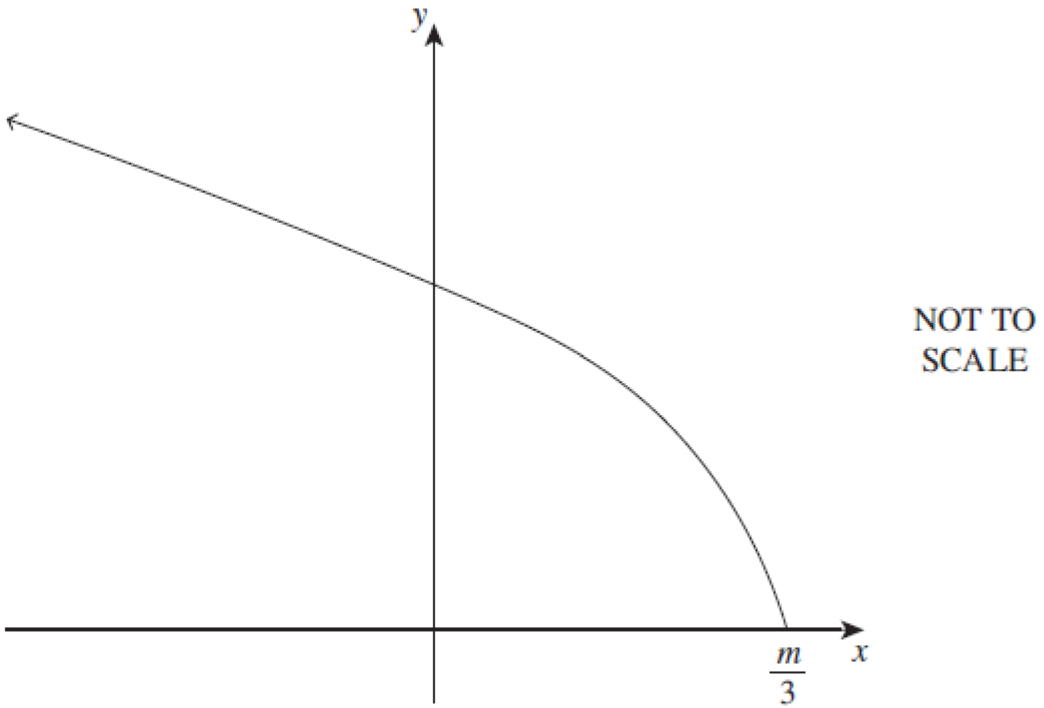
(ii) Prove that $x = \dfrac{a+b}{2}$ is a zero of $P(x)$. **1**

(iii) Prove that $P(x)$ has no stationary points. **3**

Question 14 continues on page 11

Question 14 (continued)

(c) Let $f(x)=\sqrt{m-3x}$ for $x \le \dfrac{m}{3}$. The graph of $f(x)$ is shown. **4**



The area enclosed by the graph $f(x)$, the x-axis and the y-axis is rotated about the y-axis. Find the value of m such that the volume of the solid formed is $\dfrac{5000\pi}{27}$ units^3.

End of paper

Gosford High School

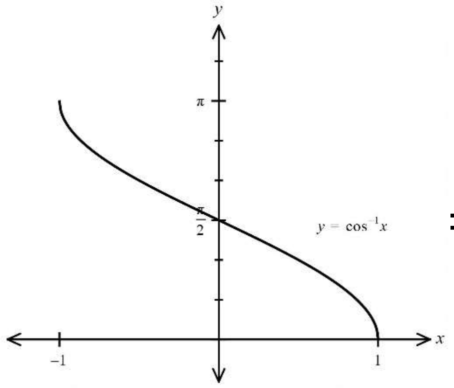
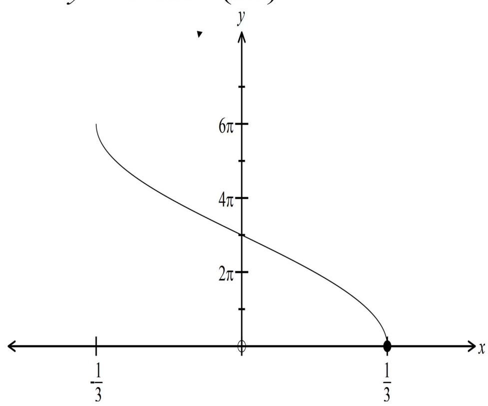
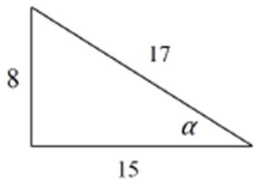
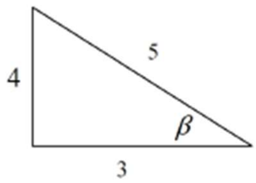
2022
TRIAL HSC
EXAMINATION

Mathematics
Extension 1

SOLUTIONS

GHS 2022 E1 Multiple Choice Worked Solutions

No	Working	Answer
1	$\frac{d}{dx}\left(\tan^{-1} x^4\right)$ $= \frac{1}{1+(x^4)^2} \times 4x^3$ $= \frac{4x^3}{1+x^8}$ Using $y' = \frac{f'(x)}{1+[f(x)]^2}$ where $f(x) = x^4$ and $f'(x) = 4x^3$	B
2	$proj_b a = \frac{5\times4+5\times(-2)}{(4^2+(-2)^2)^2}(4i-2j)$ $= 2i - j$	C
3	**A** is correct. This option is reached through a process of elimination. When $x = 1$ and $y = 1$, the gradient is 0. Therefore, we can eliminate **B** and **C**. When $x = -1$ and $y = 1$, the gradient is negative. Therefore, we can eliminate **D**, leaving **A** as the only viable option.	A
4	Let $u = (\ln x)^2$. $\therefore du = 2(\ln x)\times\frac{1}{x}dx$ *Note: Although the limits are the same for each option, the limits should always be changed when using the substitution method.* When $x = e$, $u = (\ln e)^2$ $= 1$ When $x = e^2$, $u = (\ln e^2)^2$ $= 4$ Rewriting the integral gives: $\int_e^{e^2} \frac{(\ln x)^3}{x}dx = \frac{1}{2}\int_e^{e^2} (\ln x)^2 \frac{2(\ln x)}{x}dx$ $\therefore \frac{1}{2}\int_1^4 u\, du$	A
5	$V = \pi\int_0^{\frac{\pi}{3}} \sin^2 x\, dx$ $= \frac{\pi}{2}\int_0^{\frac{\pi}{3}} 1 - \cos 2x\, dx$ Removed question due to mistake.	B

6	The girls can be chosen in $^{10}\mathbf{C}_2$ ways The boys can be chosen in $^{15}\mathbf{C}_3$ ways The number of ways the group is chosen $= {}^{10}\mathbf{C}_2 \times {}^{15}\mathbf{C}_3$ $= 20\ 475$	**D**
7	$y \cos^{-1}(x)$  ⇒ $y = 6\cos^{-1}(3x)$  Domain $\left[-\frac{1}{3}, \frac{1}{3}\right]$; Range $[0, 6\pi]$.	**C**
8	$x = \frac{t}{2}$ ① $y = 3t^2$ ② From ① $t = 2x$ ③ sub ③ into ② $y = 3 \times (2x)^2$ $y = 12x^2$	**D**
9	$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ $= \frac{8}{17}\times\frac{3}{5}+\frac{15}{17}\times\frac{4}{5}$ $= \frac{24}{85}+\frac{60}{85}$ $= \frac{84}{85}$  	**B**

10	$2\cos x - 3\sin x = R\cos(x+\theta)$ $= R\cos x\cos\theta - R\sin x\sin\theta$ Equating both sides: $2\cos x = R\cos x\cos\theta$ and $3\sin x = R\sin x\sin\theta$ Therefore: $R\cos\theta = 2 \quad (1)$ $R\sin\theta = 3 \quad (2)$ $\frac{(2)}{(1)}$ $\therefore \tan\theta = \frac{3}{2}$	**D**
Markers Comments		

Trial HSC Examination
2022

Mathematics Extension 1

Name ______________________ Teacher ________________

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample:	$2 + 4 =$	(A) 2	(B) 6	(C) 8	(D) 9
		A ○	B ●	C ○	D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

A ~~●~~ B ~~●~~ correct C ○ D ○

1.	A ○	B ●	C ○	D ○
2.	A ○	B ○	C ●	D ○
3.	A ●	B ○	C ○	D ○
4.	A ●	B ○	C ○	D ○
5.	A ○	B ●	C ○	D ○
6.	A ○	B ○	C ○	D ●
7.	A ○	B ○	C ●	D ○
8.	A ○	B ○	C ○	D ●
9.	A ○	B ●	C ○	D ○
10.	A ○	B ○	C ○	D ●

11		GHS Ext 1 HSC 2022 Question 11 Worked Solutions	Marks	Allocation and Comments
(a)		$y'' - y' - 6y = 0$ $y = Ae^{3x} + Be^{-2x}$ $y' = 3Ae^{3x} - 2Be^{-2x}$ $y'' = 9Ae^{3x} + 4Be^{-2x}$ sub into original $9Ae^{3x} + 4Be^{-2x} - (3Ae^{3x} - 2Be^{-2x}) - 6(Ae^{3x} + Be^{-2x}) = 0$ $9Ae^{3x} + 4Be^{-2x} - 3Ae^{3x} + 2Be^{-2x} - 6Ae^{3x} - 6Be^{-2x} = 0$	2	**2 for correct solution** **1 for correct differentiations oem**
(b)		Easier Solution $P(x) = (x^2 - 1)Q(x) + (3x - 1)$ $P(x) = (x + 1)(x - 1)Q(x) + (3x - 1)$ $P(1) = 3 \times 1 - 1$ $P(1) = 2$ Longer Solution $\dfrac{P(x)}{x^2 - 1} = Q(x) + \dfrac{3x - 1}{x^2 - 1}$ $\dfrac{P(x)}{(x + 1)(x - 1)} = Q(x) + \dfrac{3x - 1}{(x + 1)(x - 1)}$ $\dfrac{P(x)}{(x - 1)} = Q(x)(x + 1) + \dfrac{3x - 1}{x - 1}$ $\dfrac{P(x)}{(x - 1)} = Q(x)(x + 1) + \dfrac{3x - 3 + 2}{x - 1}$ $\dfrac{P(x)}{(x - 1)} = Q(x)(x + 1) + \dfrac{3(x - 1) + 2}{x - 1}$ $\dfrac{P(x)}{(x - 1)} = Q(x)(x + 1) + \dfrac{3(x - 1)}{x - 1} + \dfrac{2}{x - 1}$ $\dfrac{P(x)}{(x - 1)} = Q(x)(x + 1) + 3 + \dfrac{2}{x - 1}$ $\therefore \quad \dfrac{P(x)}{(x - 1)} = [Q(x)(x + 1) + 3] + \dfrac{2}{x - 1}$ $\therefore$ the remainder $= 2$	2	**2 for correct solution** **1 for reasonable progress towards solution**

11	GHS Ext 1 HSC 2022 Question 11 Worked Solutions	Marks	Allocation and Comments
(c)	$\frac{d}{dx}\left(\log_e\left(\cos^{-1}x\right)\right)=\frac{1}{\cos^{-1}x}\times\frac{-1}{\sqrt{1-x^2}}$ $=\frac{-1}{\cos^{-1}x\sqrt{1-x^2}}$	2	**2 for correct solution** **1 for reasonable progress towards solution**
(d)	$\int_0^3 \frac{x+2}{\sqrt{x+1}}dx=\int_1^2 \frac{u^2+1}{\cancel{u}}\times 2\cancel{u}\,du$ $=2\left[\frac{u^3}{3}+u\right]_1^2$ $=2\times\left[\left(\frac{2^3}{3}+2\right)-\left(\frac{1^3}{3}+1\right)\right]$ $=\frac{20}{3}$ $u^2=x+1$ $2u=\frac{dx}{du}$ $2u\,du=dx$ When $x=0, u=1$ and when $x=3, u=2$.	3	**3 for correct solution** **2 for correct substitution and reasonable progress towards the correct integration** **1 for attempting to switch variables using the given substitution**
(e)	b) $\frac{2x}{x-1}\geq 1$ $\quad x\neq 1$ $(x-1)^2\frac{2x}{x-1}\geq(x-1)^2$ $2x(x-1)\geq(x-1)^2$ $2x(x-1)-(x-1)^2\geq 0$ $(x-1)\left[2x-(x-1)\right]\geq 0$ $(x-1)(x+1)\geq 0$ Sketch $y=(x-1)(x+1)$ $\therefore$, $x>1$	3	**3 for correct solution** **2 for $(x-1)(x+1)\geq 0$ but not recognising that $x\neq 1$** **1 for some correct progress made**

11		GHS Ext 1 HSC 2022 Question 11 Worked Solutions	Marks	Allocation and Comments
(f)		$f'(x) = -\dfrac{3}{\sqrt{16-9x^2}}$ $f(x) = \displaystyle\int -\dfrac{3\,dx}{\sqrt{16-9x^2}}$ $f(x) = -\displaystyle\int \dfrac{3\,dx}{\sqrt{4^2-(3x)^2}}$ $= -\sin^{-1}\dfrac{3x}{4} + c$ when $x = 0$, $f(x) = -\dfrac{\pi}{2}$ $-\dfrac{\pi}{2} = -\sin^{-1}(0) + c$ $-\dfrac{\pi}{2} = c$ $f(x) = -\sin^{-1}\dfrac{3x}{4} - \dfrac{\pi}{2}$ Answer can also be found to be: $f(x) = \cos^{-1}\dfrac{3x}{4} - \pi$	2	**2 for correct solution for F(x)** **1 for correct integration but did not find c or c is incorrect**

Markers Comments

Part a was done very well. Students who struggled generally differentiated the e function incorrectly or tried to manipulate the formula instead of just substituting in the y’ and y’’.

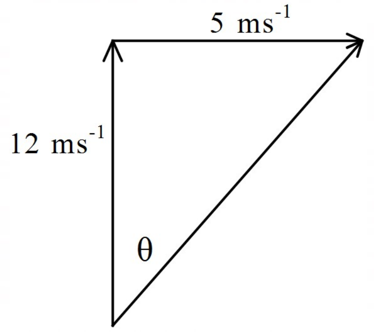
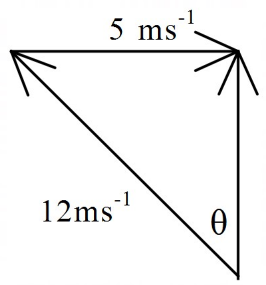
Part b was done quite poorly. Students who remembered that P(x) = A(x)Q(x) + R(x) were easily able to substitute in using the remainder theorem and find a remainder of 2. Other students when not given a P(x) or Q(x) attempted to use division or construct an original polynomial, which was not successful.

Part c was also done very well. Most students recognised the correct division. Poor answers put the $\cos^{-1} x$ in the numerator due to confusion with fractions on fractions.

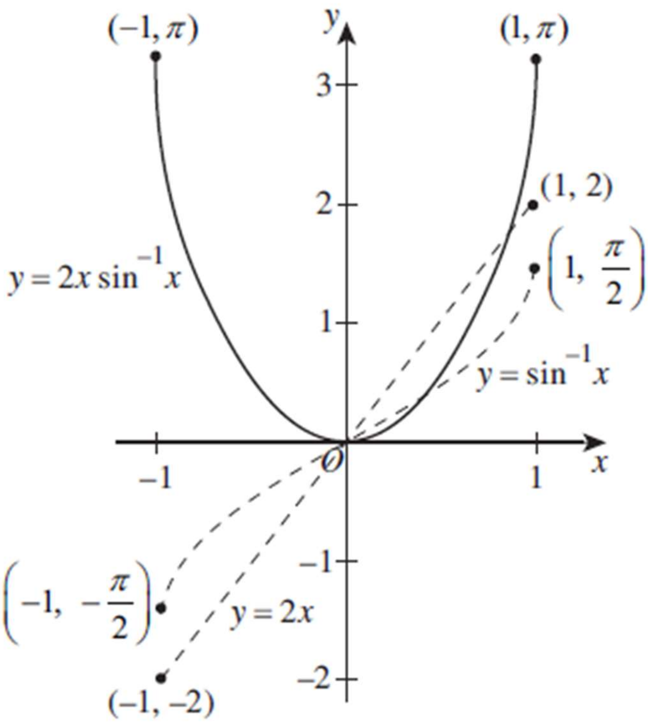
Part d was done fairly poorly. Most students attempted to substitute something but didn’t correctly change all relevant fields: bounds, dx and f(x). Once substitutions were complete (either correctly or incorrectly) most students integrated and substituted confidently.

Part e, most students didn’t recognise the need to multiply by the square of the denominator.

Part f, students who remembered or used their formula sheet to recognise the inverse trig integration generally did well. Substitution was fairly successful.

12		**GHS Ext 1 HSC 2022** **Question 12 Worked Solutions**	Marks	**Allocation and Comments**
(a)	(i)	$\underset{\sim}{v} = 5\underset{\sim}{i} + 12\underset{\sim}{j}$ $\lvert\underset{\sim}{v}\rvert = \sqrt{5^2 + 12^2}$ $= 13\text{ms}^{-1}$  	2	**2 for correct solution giving both component vector and magnitude** **1 for either the component vector or magnitude correct**
	(ii)	$\tan\theta = \dfrac{5}{12}$ $\theta = 22°37'11.51''$ $= 023°T$ $\dfrac{x}{13} = \dfrac{120}{12}$ $x = \dfrac{120 \times 13}{12}$ $x = 130\text{m}$	2	**2 for correct solution giving both the bearing and the distance** **1 for either correct bearing or distance**
	(iii)	 $\sin\theta = \dfrac{5}{12}$ $\theta = 24°37'$ $\therefore$ Bearing $= 360° - 24°37'$ $= 335°$ (nearest degree)	2	**2 for correct solution giving the bearing** **1 for the correct angle but not converting to a bearing** **0 marks awarded if students attempted to use the 23° answer from (ii)**

12		GHS Ext 1 HSC 2022 Question 12 Worked Solutions	Marks	Allocation and Comments
(b)		$\int \sin^2 x \cos^2 x dx$ $\int (\sin x \cos x)^2 \, dx$ $\qquad \sin 2x = 2\sin x \cos x$ $= \int \left(\frac{\sin 2x}{2}\right)^2 dx$ $= \int \frac{1}{4} (\sin 2x)^2 \, dx$ $\qquad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $= \int \frac{1}{4} \times \frac{1}{2} [1 - \cos(2 \times 2x)] \, dx$ $= \frac{1}{8} \int (1 - \cos 4x) dx$ $= \frac{1}{8} \left[x - \frac{1}{4} \sin 4x\right] + c$ $= \frac{1}{8} x - \frac{1}{32} \sin 4x + c$	2	**2 for correct solution** **1 for rewriting and using identity to simplify or some progress towards finding the integral**
(c)	(i)	$f(x) = 2x \sin^{-1} x$ $f(-x) = 2(-x)\sin^{-1}(-x)$ $= -2x \sin^{-1}(-x)$ Since the function is odd, $\sin^{-1}(-x) = -\sin^{-1} x$. $f(-x) = 2x \sin^{-1} x$ $= f(x)$ Therefore, $f(x)$ is an even function.	1	**1 for correct solution**

<table>
<tr><td colspan="2">12</td><td>GHS Ext 1 HSC 2022
Question 12 Worked Solutions</td><td>Marks</td><td>Allocation and Comments</td></tr>
<tr><td></td><td>(ii)</td><td>
</td><td>2</td><td>2 for correct sketch of the function showing all important features

1 for providing coordinates of the endpoints and/or providing a sketch with some minor errors</td></tr>
</table>

(d)		Step 1: Proving the statement is true for $n = 1$ gives: $\text{LHS} = 1\times 2^{-(1-1)}$ $= 1$ $\text{RHS} = \dfrac{2^{1+1}-1-2}{2^{1-1}}$ $= \dfrac{4-1-2}{1}$ $= 1$ $\text{LHS} = \text{RHS}$ Therefore, the statement is true for $n = 1$. Step 2: Assuming the statement is true for $n = k$ gives: $1+2\times 2^{-1}+3\times 2^{-2}+4\times 2^{-3}$ $+\ldots+k\times 2^{-(k-1)} = \dfrac{2^{k+1}-k-2}{2^{k-1}}$ Step 3: Proving that the statement is true for $n = k+1$ gives: $1+2\times 2^{-1}+3\times 2^{-2}+4\times 2^{-3}+\ldots+k\times 2^{-(k-1)}$ $+(k+1)\times 2^{-k} = \dfrac{2^{k+2}-(k+1)-2}{2^{k}}$ $= \dfrac{2^{k+2}-k-3}{2^{k}}$ $\text{LHS} = 1+2\times 2^{-1}+3\times 2^{-2}+4\times 2^{-3}+\ldots$ $+k\times 2^{-(k-1)}+(k+1)\times 2^{-k}$ $= \dfrac{2^{k+1}-k-2}{2^{k-1}}+(k+1)\times 2^{-k}$ (by assumption) $= \dfrac{2^{k+1}-k-2}{2^{k-1}}+\dfrac{k+1}{2^{k}}$ $= \dfrac{2^{k+2}-2k-4}{2^{k}}+\dfrac{k+1}{2^{k}}$ $= \dfrac{2^{k+2}-2k-4+k+1}{2^{k}}$ $= \dfrac{2^{k+2}-k-3}{2^{k}}$ $= \text{RHS}$ If $n = k$ is true, then $n = k+1$ is true. Therefore, by mathematical induction, the statement is true for $n \geq 1$.	3	**3 for correct solution including final statement** **2 for progress towards proving (k+1)** **1 for showing steps 1 & 2 and substituting (k+1)**

12	GHS Ext 1 HSC 2022 Question 12 Worked Solutions	Marks	Allocation and Comments

Markers Comments

(a) This question was mostly well attempted. Some students missed details in the questions requiring them to find a second piece of information. In part (ii) students were required to find a bearing. Responses had to be in a convention form – either a three figure bearing 023° or a compass bearing N23°E to receive this mark.

(b) Few students used the identity $\sin(x)\cos(x) = \frac{1}{2}\sin 2x$. Most students used inefficient methods to convert the function to a recognisable form for integration and in the process made algebraic errors preventing them from obtaining the correct solution.

(c) (i) When showing that f(x) is even, better responses should have an initial line that shows substitution of (-x) and then show how it simplifies to give f(x). (ii) despite the previous part of the question indicating that f(x) is an even function, many responses sketched an odd function. Better responses included sketches of y=2x, y=$\sin^{-1}$x and then sketched the product of functions to obtain the required function.

(d) Few responses demonstrated thorough knowledge of the process of proving by mathematical induction. When showing that the statement is true for the initial case, students must show the substitution into each side and show that they simplify to the same value. The first step should NOT be 1=1 or equivalent. Refer to solutions as a guide about structuring this line. For the assumption and the next integer case, the left hand side should be a sum of terms from n=1 to n=k (or n=k+1). Too many responses simply wrote the last case which is incorrect.

13		GHS Ext 1 HSC 2022 Question 13 Worked Solutions	Marks	Allocation and Comments
(a)	(i)	$\cos(A-B)$ $= \cos A\cos B + \sin A \sin B$ $= \cos A \cos B\left(1 + \frac{\sin A \sin B}{\cos A \cos B}\right)$ $= \cos A \cos B\,(1 + \tan A \tan B)$	1	**1 for correct solution**

13	**GHS Ext 1 HSC 2022** **Question 13 Worked Solutions**	Marks	**Allocation and Comments**

Markers Comments

Better responses provided a link between the given result for Cos(A – B) on the reference sheet, and the result they were asked to solve.

Students need to convince the marker, they would have got the result if it was not given to them in the question. Hence the need for extra working steps.

(ii)	$\cos(A-B) = \cos A \cos B\,(1+\tan A \tan B)$ $= \cos A \cos B\,(1+(-1))$ $= 0$ $\therefore A - B = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \cdots = (2k+1)\frac{\pi}{2}$ Since $A < \pi$ and $B > 0$, $\therefore A - B < \pi$ $\therefore A - B = \frac{\pi}{2}$	1	**1 for correct solution**

Markers Comments

Most responses assumed that when Cos(A – B) = 0, then A – B = Pi/2 and did not consider other possible results.

Do not substitute in values to prove the general result.

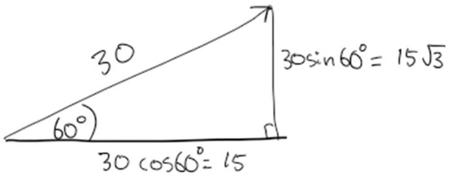
13		GHS Ext 1 HSC 2022 Question 13 Worked Solutions	Marks	Allocation and Comments
(b)		$\sqrt{3}\sin\theta+\cos\theta-\sqrt{3}=0$ $\sqrt{3}\sin\theta+\cos\theta=\sqrt{3}$ $\sqrt{3}\sin\theta+\cos\theta=r\sin(\theta+\alpha)$ $=r\cos\alpha\sin\theta+r\sin\alpha\cos\theta$ Equating coefficients: $r\cos\alpha=\sqrt{3}$ $r\sin\alpha=1$ $r^2\cos^2\alpha+r^2\sin^2\alpha=\left(\sqrt{3}\right)^2+1^2$ $r^2\left(\cos^2\alpha+\sin^2\alpha\right)=4$ $r^2=4$ $r=2\ \ (\text{since } r>0)$ and $\dfrac{\sin\alpha}{\cos\alpha}=\dfrac{1}{\sqrt{3}}$ $\tan\alpha=\dfrac{1}{\sqrt{3}}$ $\alpha=\dfrac{\pi}{6}$ $2\sin\left(\theta+\dfrac{\pi}{6}\right)=\sqrt{3}$ $\sin\left(\theta+\dfrac{\pi}{6}\right)=\dfrac{\sqrt{3}}{2}$ $\theta+\dfrac{\pi}{6}=\dfrac{\pi}{3},\dfrac{2\pi}{3},\dfrac{13\pi}{3},\ldots$ $\theta=\dfrac{\pi}{6},\dfrac{\pi}{2},\dfrac{25\pi}{6},\ldots \text{ but } 0\le\theta\le 2\pi$ $\therefore \theta=\dfrac{\pi}{6} \text{ or } \dfrac{\pi}{2}$	3	**3 for correct solution** **2 for progress towards correct answers** **1 for changing the form of the problem using the auxiliary equations**

Markers Comments:

Better responses knew how to use the Auxiliary Angle Method rather than a variety of other methods.

Those that used the Sin result tended to get the question all correct.

Those that used the Cos result, tended to forget to check for initial angles outside of the first quadrant, i.e. negative angles in the 4th quadrant, which lead to the second result of Pi/6.

13		GHS Ext 1 HSC 2022 **Question 13 Worked Solutions**	Marks	**Allocation and Comments**
(c)		 $\ddot{x} = 0$ $\dot{x} = C$ when $t=0$, $\dot{x} = 15$ $(C=15)$ $\therefore \dot{x} = 15$ $x = 15t + D$ when $t=0$, $x=0$ $\therefore D = 0$ $\therefore x = 15t$ $\ddot{y} = -g$ $\ddot{y} = -10$ $\dot{y} = -10t + E$ when $t=0$, $\dot{y} = 15\sqrt{3}$ $\therefore E = 15\sqrt{3}$ $\Rightarrow \dot{y} = -10t + 15\sqrt{3}$ $y = -5t^2 + 15\sqrt{3}t + F$ when $t=0$, $y=0 \Rightarrow F = 0$ $\therefore y = -5t^2 + 15\sqrt{3}t$ $\therefore \underset{\sim}{r}(t) = 15t\underset{\sim}{i} + (15\sqrt{3}t - 5t^2)\underset{\sim}{j}$	3	**3 for correct solution** **2 for finding the equations of motion in either the horizontal OR vertical direction AND makes some progress in the other direction** **1 for making some progress in deriving the horizontal OR vertical equation of motion**

Markers comments:

Since the question asked students "By considering the equations….." then students needed to show that they started or at least referenced them in their solution. Many students took too many shortcuts.

Students need to recognise 30 degrees and 60 degrees give exact values, so they are expected to use them and not convert to decimals.

(d)		$\int \frac{dy}{\sqrt{1-y^2}} = \int dx$ ………. Line 1 $sin^{-1}y = x + c$ ……….Line 2 $y = sin\,(x + c)$ $-\frac{1}{2} = sin\,(\frac{5\pi}{6} + C)$ $-\frac{\pi}{6} = \frac{5\pi}{6} + C$ $C = \pi$ $y = sin\,(x - \pi) \qquad -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$	3	**3 for correct solution** **2 for significant progress** **1 for some progress**

<table>
<tr><td>13</td><td colspan="2">GHS Ext 1 HSC 2022
Question 13 Worked Solutions</td><td>Marks</td><td>Allocation and Comments</td></tr>
<tr><td colspan="5">Markers comments:
Since the question involved an integration, leading to Inverse Sin, then students need to remember that the domain for y is restricted.
Check the Reference Sheet as the integrals were straight forward.
Be careful using brackets as some students wrote y=sin(x) + c instead of y=sin(x + c).</td></tr>
<tr><td>(e)</td><td></td><td>$$\begin{aligned} V &= \pi \int_0^{\pi} 9\cos^2 \frac{x}{2}\, dx \\ &= 9\pi \int_0^{\pi} \left(\frac{1}{2} + \frac{1}{2}\cos x\right)\, dx \\ &= \frac{9\pi}{2}\left[x + \sin x\right]_0^{\pi} \\ &= \frac{9\pi}{2}\left((\pi + \sin\pi) - (0+0)\right) \\ &= \frac{9\pi^2}{2} \end{aligned}$$</td><td>3</td><td>3 for correct solution

2 for correct volume integral and primitive function

1 for correct volume integral</td></tr>
<tr><td colspan="5">Markers Comments
Generally well done. Don’t forget to square the function and multiply by Pi.
Students who got a correct answer from incorrect working, did not receive full marks.</td></tr>
</table>

14		GHS Ext 1 HSC 2022 Question 14 Worked Solutions	Marks	Allocation and Comments
(a)	(i)	$\frac{dT}{dt} = k(200 - T)$ $\frac{dT}{200 - T} = kdt$ $\int \frac{dT}{200 - T} = \int kdt$ $-\ln\lvert 200 - T\rvert = kt + c$ $\ln\lvert 200 - T\rvert = -kt - c$ $200 - T = \pm e^{-c} e^{-kt}$ $T = 200 \pm e^{-c} e^{-kt}$ Let $A = \pm e^{-c}$ $\therefore T = 200 + Ae^{-kt}$	1	**2 for correct solution** **1 for making progress towards solving the differential equation.** **ie placing the correct variables on each side of the equation and trying to integrate.** *Marker's Comment* Many students could not identify a differential equation. The question asked you to show the expression for T starting from the differential equation (as shown in the solution). Marks were not awarded for differentiating the expression for T as you were not asked to use T to show the differential equation.

14		GHS Ext 1 HSC 2022 Question 14 Worked Solutions	Marks	Allocation and Comments
	(ii)	When $t = 0 \quad T = 0°C$ $0 = 200 + A$ $A = -200$ $\therefore \quad T = 200 - 200e^{-kt}$ When $t = 10 \quad T = 20°C$ $20 = 200 - 200e^{-10k}$ $-\dfrac{180}{-200} = e^{-10k}$ $\dfrac{9}{10} = e^{-10k}$ $\dfrac{9}{10} = \dfrac{1}{e^{10k}}$ $\dfrac{10}{9} = e^{10k}$ $\ln\left(\dfrac{10}{9}\right) = 10k$ $k = \dfrac{1}{10}\ln\left(\dfrac{10}{9}\right)$	2	**2 for correct solution** **1 for correctly finding *A* and making some progress to finding *k*** *Marker's Comment* Generally well answered.

	(iii)	When $T = 60°C$ $T = 200 - 200\,e^{-\left(\frac{\ln\left(\frac{10}{9}\right)}{10}\right)t}$ $60 = 200 - 200\,e^{-\left(\frac{\ln\left(\frac{10}{9}\right)}{10}\right)t}$ $-140 = -200\,e^{-\left(\frac{\ln\left(\frac{10}{9}\right)}{10}\right)t}$ $\frac{-140}{-200} = e^{-\left(\frac{\ln\left(\frac{10}{9}\right)}{10}\right)t}$ $\frac{7}{10} = e^{-\left(\frac{\ln\left(\frac{10}{9}\right)}{10}\right)t}$ $\ln\left(\frac{7}{10}\right) = -\left(\frac{\ln\left(\frac{10}{9}\right)}{10}\right)t$ $t = \frac{\ln\left(\frac{7}{10}\right)}{-\left(\frac{\ln\left(\frac{10}{9}\right)}{10}\right)t}$ $t = \frac{10\left(\ln\left(\frac{7}{10}\right)\right)}{-\left(\ln\left(\frac{10}{9}\right)\right)}$ $t = 33.85$ mins $t = 34$ mins ∴ It will take $34 - 10 = 24$ mins longer	2	**2 for correct solution** **Students can change k to a decimal and no penalty should be made for rounding errors** **1 for correct substitution and some progress towards the answer** *Marker's Comment* Well answered but students should be mindful to read the question carefully. Many students were not awarded full marks because they overlooked the question of "how much longer".

14		GHS Ext 1 HSC 2022 Question 14 Worked Solutions	Marks	Allocation and Comments
(b)	(i)	i. (1 mark) $P(x) = (x-a)^3 + (x-b)^2$ $P(b) = -8$ $\therefore (b-a)^3 + (b-b)^2 = -8$ $(b-a)^3 = -8$ $b - a = -2 \qquad (\dagger)$ Applying the remainder theorem and evaluating $P(a)$: $P(a) = (a-a)^3 + (a-b)^2$ $= (a-b)^2$ $= 4$ Hence the remainder when divided by $(x-a)$, is 4.	2	**2 for correct solution** **1 for finding $b - a = -2$** *Marker's Comment* Many students did not use the remainder theorem and the information provided as a starting point. i.e. find $P(b)$. They could then substitute the answer into $P(a)$
	(ii)	ii. (1 mark) $P\left(\frac{a+b}{2}\right) = \left(\frac{a+b}{2} - a\right)^3 + \left(\frac{a+b}{2} - b\right)^2$ $= \left(\frac{a+b-2a}{2}\right)^3 + \left(\frac{a+b-2b}{2}\right)^2$ $= \left(\frac{b-a}{2}\right)^3 + \left(\frac{a-b}{2}\right)^2$ $= \left(-\frac{2}{2}\right)^3 + \left(\frac{2}{2}\right)^2$ $= -1 + 1 = 0$ Hence $x = \frac{a+b}{2}$ is a zero of $P(x)$.	1	**1 for correct solution** *Marker's Comment* Poorly answered. Many students should substitute $P\left(\frac{a+b}{2}\right)$ and simplify the expression.

14		GHS Ext 1 HSC 2022 Question 14 Worked Solutions	Marks	Allocation and Comments
	(iii)	$P(x) = (x-a)^3 + (x-b)^2$ Differentiating, $P'(x) = 3(x-a)^2 + 2(x-b)$ Stationary points occur when $P'(x) = 0$: $3(x-a)^2 + 2(x-b) = 0$ $3x^2 - 6ax + 3a^2 + 2x - 2b = 0$ $3x^2 + (2-6a)x + (3a^2 - 2b) = 0$ Checking the discriminant of this quadratic: $\Delta = (2-6a)^2 - 4(3)(3a^2 - 2b)$ $= 4 - 24a + 36a^2 - 36a^2 + 24b$ $= 4 - 24(a-b)$ $= 4 - 24(2)$ < 0 $P'(x) = 0$ has no real roots. Hence $P(x)$ has no stationary points.	3	✓ [1] for obtaining $3(x-a)^2 + 2(x-b) = 0$. ✓ [1] for correct substitution into Δ. ✓ [1] for correct proof. *Marker's Comment* Many students differentiated the expression but were unable to use previous parts to demonstrate there are no real roots. Students should be mindful to use previous parts in a multi-part question.

(c)		To find the y-intercept, let $x=0$. $y=\sqrt{m-3(0)}$ $=\sqrt{m}$ Rearranging to make x the subject gives: $y=\sqrt{m-3x}$ $y^2=m-3x$ $3x=m-y^2$ $x=\frac{1}{3}(m-y^2)$ $\text{volume}=\pi\int_0^{\sqrt{m}}\left(\frac{1}{3}(m-y^2)\right)^2 dy$ $=\frac{\pi}{9}\int_0^{\sqrt{m}}(m-y^2)^2 dy$ $=\frac{\pi}{9}\int_0^{\sqrt{m}} m^2-2my^2+y^4 dy$ $=\frac{\pi}{9}\left[m^2y-\frac{2my^3}{3}+\frac{y^5}{5}\right]_0^{\sqrt{m}}$ $=\frac{\pi}{9}\left(m^2\times\sqrt{m}-\frac{2m\times(\sqrt{m})^3}{3}+\frac{(\sqrt{m})^5}{5}-(0-0+0)\right)$ $=\frac{\pi}{9}\left(m^2\sqrt{m}-\frac{2}{3}m^2\sqrt{m}+\frac{1}{5}m^2\sqrt{m}\right)$ $=\frac{\pi}{9}\times\frac{8}{15}m^2\sqrt{m}$ $=\frac{8\pi}{135}m^2\sqrt{m}$ $\frac{8\pi}{135}m^2\sqrt{m}=\frac{5000\pi}{27}$ $m^2\sqrt{m}=3125$ $m^{\frac{5}{2}}=3125$ $m=(3125)^{\frac{2}{5}}$ $=25$	4	**4 for correct solution** **3 for finding the integrand for the volume of solid of revolution** **2 for finding the y-intercept** **AND** **rearranging the equation to make x the subject** **1 for finding the y-intercept** **OR** **rearranging the equation to make x the subject** *Marker's Comment* Many students did not rotate around the *y* axis. These solutions attracted no marks as the calculus was made far easier than the correct technique. This is a tough question and students should be careful not to make a careless error. Students could achieve this by setting out their work as neatly as possible to avoid arithmetic errors.